

Forward Self-Similar Solutions of the Navier-Stokes Equations in the Half Space

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Abstract. For the incompressible Navier-Stokes equations in the 3D half space, we show the existence of forward self-similar solutions for arbitrarily large self-similar initial data.

1 Introduction

Let $\mathbb{R}_+^3 = \{x = (x_1, x_2, x_3) : x_3 > 0\}$ be a half space with boundary $\partial\mathbb{R}_+^3 = \{x = (x_1, x_2, 0)\}$. Consider the 3D incompressible Navier-Stokes equations for velocity $u : \mathbb{R}_+^3 \times [0, \infty) \rightarrow \mathbb{R}^3$ and pressure $p : \mathbb{R}_+^3 \times [0, \infty) \rightarrow \mathbb{R}$,

$$\partial_t u - \Delta u + (u \cdot \nabla)u + \nabla p = 0, \quad \operatorname{div} u = 0, \quad (1.1)$$

in $\mathbb{R}_+^3 \times [0, \infty)$, coupled with the boundary condition

$$u|_{\partial\mathbb{R}_+^3} = 0, \quad (1.2)$$

and the initial condition

$$u|_{t=0} = a, \quad \operatorname{div} a = 0, \quad a|_{\partial\mathbb{R}_+^3} = 0. \quad (1.3)$$

The system (1.1) enjoys a scaling property: If $u(x, t)$ is a solution, then so is

$$u^{(\lambda)}(x, t) := \lambda u(\lambda x, \lambda^2 t) \quad (1.4)$$

for any $\lambda > 0$. We say that $u(x, t)$ is **self-similar** (SS) if $u = u^{(\lambda)}$ for every $\lambda > 0$. In that case,

$$u(x, t) = \frac{1}{\sqrt{2t}} U\left(\frac{x}{\sqrt{2t}}\right), \quad (1.5)$$

where $U(x) = u(x, \frac{1}{2})$. It is called **discretely self-similar** (DSS) if $u = u^{(\lambda)}$ for one particular $\lambda > 1$. To get self-similar solutions $u(x, t)$ we usually assume the initial data $a(x)$ is also self-similar, i.e.,

$$a(x) = \frac{a(\hat{x})}{|\hat{x}|}, \quad \hat{x} = \frac{x}{|x|}. \quad (1.6)$$

In view of the above, it is natural to look for solutions satisfying

$$|u(x, t)| \leq \frac{C(C_*)}{|x|}, \quad \text{or} \quad \|u(\cdot, t)\|_{L^{q,r}} \leq C(C_*), \quad (1.7)$$

where C_* is some norm of the initial data a . By $L^{q,r}$, $1 \leq q, r \leq \infty$, we denote the Lorentz spaces. In such classes, with sufficiently small C_* , the unique existence of mild solutions –

solutions of the integral equation version of (1.1)–(1.3) via contraction mapping argument, – has been obtained by Giga-Miyakawa [6] and refined by Cannone-Meyer-Planchon [4, 5]. In the content of the half space (and smooth exterior domains), it follows from Yamazaki [25]. As a consequence, if $a(x)$ is SS or DSS with small norm C_* and $u(x, t)$ is a corresponding solution satisfying (1.7) with small $C(C_*)$, the uniqueness property ensures that $u(x, t)$ is also SS or DSS, because $u^{(\lambda)}$ is another solution with the same bound and same initial data $a^{(\lambda)} = a$. For large C_* , mild solutions still make sense but there is no existence theory since perturbative methods like the contraction mapping no longer work.

Alternatively, one may try to extend the concept of weak solutions (which requires $u_0 \in L^2(\mathbb{R}^3)$) to more general initial data. One such theory is local-Leray solutions in L^2_{loc} , constructed by Lemarié-Rieusset [18]. However, there is no uniqueness theorem for them and hence the existence of large SS or DSS solutions was unknown. Recently, Jia and Šverák [7] constructed SS solutions for every SS u_0 which is locally Hölder continuous. Their main tool is a local Hölder estimate for local-Leray solutions near $t = 0$, assuming minimal control of the initial data in the large. This estimate enables them to prove a priori estimates of SS solutions, and then to show their existence by the Leray-Schauder degree theorem. This result is extended by Tsai [24] to the existence of discretely self-similar solutions.

When the domain is the half space \mathbb{R}^3_+ , however, there is so far no analog theory of local-Leray solutions. Hence the method of [7], [24] is not applicable.

In this note, our goal is to construct SS solutions in half space for arbitrary large data. By BC_w we denote bounded and weak-* continuous functions. Our main theorem is the following.

Theorem 1.1. *Let $\Omega = \mathbb{R}^3_+$ and A be the Stokes operator in Ω . For any self-similar vector field $a \in C^1_{loc}(\bar{\Omega} \setminus \{0\})$ satisfying $\operatorname{div} a = 0$, $a|_{\partial\Omega} = 0$, there is a smooth self-similar mild solution $u \in BC_w([0, \infty); L^{3,\infty}_\sigma(\Omega))$ of (1.1) with $u(0) = a$ and*

$$\|u(t) - e^{-tA}a\|_{L^2(\Omega)} = Ct^{1/4}, \quad \|\nabla(u(t) - e^{-tA}a)\|_{L^2(\Omega)} = Ct^{-1/4}, \quad \forall t > 0. \quad (1.8)$$

Comments on Theorem 1.1

1. There is no restriction on the size of a .
2. It is concerned only with existence. There is no assertion on uniqueness.
3. Our approach also gives a second construction of large self-similar solutions in the whole space \mathbb{R}^3 , but for initial data more restrictive (C^1) than those of [7]. In fact, it would show the existence of self-similar solutions in the cones

$$K_\alpha = \{0 \leq \phi \leq \alpha\}, \quad (0 < \alpha \leq \pi),$$

(in spherical coordinates), if one could verify Assumption 3.1 for $e^{-\frac{1}{2}A}a$. We are able to verify it only for $\alpha = \pi/2$ and $\alpha = \pi$.

4. We have the uniform bound (1.7) for $u_0(t) = e^{-tA}a$ and we will show $|u_0(x, t)| \lesssim (\sqrt{t} + |x|)^{-1}$ in Section 6. We expect $u_0(t) \notin L^q(\Omega)$ for any $q \leq 3$, and $\|u_0(t)\|_{L^q} \rightarrow \infty$ as $t \rightarrow 0_+$ for $q > 3$. The difference $v = u - u_0$ is more localized: by interpolating (1.8), $\|v(t)\|_{L^q} \rightarrow 0$ as $t \rightarrow 0_+$ for all $q \in [2, 3]$. Although $\|v(t)\|_{L^3(\Omega)} = C$ for $t > 0$, $v(t)$ weakly converges to 0 in L^3 as $t \rightarrow 0_+$, as easily shown by approximating the test function by $L^2 \cap L^{3/2}$ functions. Both $u_0(t)$ and $v(t)$ belong to $L^\infty(\mathbb{R}_+; L^{3,\infty}(\mathbb{R}^3_+))$.

We now outline our proof. Unlike previous approaches based on the evolution equations, we directly prove the existence of the profile U in (1.5). It is based on the a priori estimates for U using the classical Leray-Schauder fixed point theorem and the Leray *reductio ad absurdum* argument (which has been fruitfully applied in recent papers of Korobkov, Pileckas and Russo [11]–[15] on the boundary value problem of stationary Navier-Stokes equations). Specifically, the profile $U(x)$ satisfies the Leray equations

$$-\Delta U - U - x \cdot \nabla U + (U \cdot \nabla)U + \nabla P = 0, \quad \operatorname{div} U = 0 \quad (1.9)$$

in \mathbb{R}_+^3 with zero boundary condition and, in a suitable sense,

$$U(x) \rightarrow U_0(x) := (e^{-\frac{1}{2}A}a)(x) \quad \text{as } |x| \rightarrow \infty. \quad (1.10)$$

System (1.9) was proposed by Leray [17], with the opposite sign for $U + x \cdot \nabla U$, for the study of singular *backward* self-similar solutions of (1.1) in \mathbb{R}^3 of the form $u(x, t) = \frac{1}{\sqrt{-2t}} U\left(\frac{x}{\sqrt{-2t}}\right)$. Their triviality with $U \in L^q(\mathbb{R}^3)$, $3 \leq q \leq \infty$, was established in [19] and [23]. In the forward case and in the whole space setting, we have (see [7, 24])

$$U_0(x) \sim |x|^{-1}, \quad V(x) := U(x) - U_0(x) \lesssim |x|^{-2} \quad (|x| > 1). \quad (1.11)$$

In the half space setting, it is not clear if one can show pointwise decay bound for V . We will however show that $V(x)$ is a priori bounded in $H_0^1(\mathbb{R}_+^3)$, and use this a priori bound to construct a solution. Due to lack of compactness of H_0^1 at spatial infinity, we will use the *invading method*, introduced by Leray [16]: We will approximate $\Omega = \mathbb{R}_+^3$ by $\Omega_k = \Omega \cap B_k$, $k = 1, 2, 3, \dots$, where B_k is an increasing sequence of concentric balls, construct solutions V_k in Ω_k of the difference equation (3.1) with zero boundary condition, and extract a subsequence converging to a desired solution V in \mathbb{R}_+^3 .

Our proof is structured as follows. We will first recall some properties for Euler flows in Section 2, and then use it to show that V_k are uniformly bounded in $H_0^1(\Omega_k)$ in Section 3. In Section 4, we construct V_k using the a priori bound and a linear version of the Leray-Schauder theorem, and extract a weak limit V using the uniform bound. The arguments in Sections 2–4 are valid as long as one can show that $U_0 = e^{-\frac{1}{2}A_\Omega}a$, A_Ω being the Stokes operator in Ω , satisfies certain decay properties to be specified in Assumption 3.1. In Sections 5 we show that, for $\Omega = \mathbb{R}_+^3$ and those initial data a considered in Theorem 1.1, U_0 indeed satisfies Assumption 3.1. We finally verify that $u(x, t)$ defined by (1.5) satisfies the assertions of Theorem 1.1 in Section 6.

Because our existence proof does not use the evolution equation, we do not need the nonlinear version of the Leray-Schauder theorem as in [7, 24]. As a side benefit, we do not need to check the small-large uniqueness (cf. [24, Lemma 4.1]).

2 Some properties of solutions to the Euler system

For $q \geq 1$ denote by $D^{k,q}(\Omega)$ the set of functions $f \in W_{\text{loc}}^{k,q}(\Omega)$ such that $\|f\|_{D^{k,q}(\Omega)} = \|\nabla^k f\|_{L^q(\Omega)} < \infty$. Recall, that by Sobolev Embedding Theorem, if $qk < n$, then for any $f \in D^{k,q}(\mathbb{R}^n)$ there exists a constant $c \in \mathbb{R}$ such that $f - c \in L^p(\mathbb{R}^n)$ with $p = \frac{nq}{n-kq}$. In particular,

$$f \in D^{1,2}(\mathbb{R}^3) \Rightarrow f - c \in L^6(\mathbb{R}^3); \quad f \in D^{1,3/2}(\mathbb{R}^3) \Rightarrow f - c \in L^3(\mathbb{R}^3). \quad (2.1)$$

Further, denote by $D_0^{1,2}(\Omega)$ the closure of the set of all smooth functions having compact supports in Ω with respect to the norm $\|\cdot\|_{D^{1,2}(\Omega)}$, and $H(\Omega) = \{\mathbf{v} \in D_0^{1,2}(\Omega) : \operatorname{div} \mathbf{v} = 0\}$. In particular,

$$H(\Omega) \hookrightarrow L^6(\Omega) \quad (2.2)$$

(recall, that by Sobolev inequality $\|f\|_{L^6(\mathbb{R}^3)} \leq C\|\nabla f\|_{L^2(\mathbb{R}^3)}$ holds for every function $f \in C_c^\infty(\mathbb{R}^3)$ having compact support in \mathbb{R}^3 , see [1, Theorem 4.31]).

Assume that the following conditions are fulfilled:

(E) Let Ω be a domain in \mathbb{R}^3 with (possibly unbounded) connected Lipschitz boundary $\Gamma = \partial\Omega$, and the functions $\mathbf{v} \in H(\Omega)$ and $p \in D^{1,3/2}(\Omega) \cap L^3(\Omega)$ satisfy the Euler system

$$\begin{cases} (\mathbf{v} \cdot \nabla) \mathbf{v} + \nabla p = 0 & \text{in } \Omega, \\ \operatorname{div} \mathbf{v} = 0 & \text{in } \Omega, \\ \mathbf{v} = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.3)$$

The next statement was proved in [9, Lemma 4] and in [3, Theorem 2.2].

Theorem 2.1. *Let the conditions (E) be fulfilled. Then*

$$\exists \hat{p}_0 \in \mathbb{R} : \quad p(x) \equiv \hat{p}_0 \quad \text{for } \mathfrak{H}^2 - \text{almost all } x \in \partial\Omega. \quad (2.4)$$

Here and henceforth we denote by \mathfrak{H}^m the m -dimensional Hausdorff measure, i.e., $\mathfrak{H}^m(F) = \lim_{t \rightarrow 0+} \mathfrak{H}_t^m(F)$, where $\mathfrak{H}_t^m(F) = \inf \left\{ \sum_{i=1}^{\infty} (\operatorname{diam} F_i)^m : \operatorname{diam} F_i \leq t, F \subset \bigcup_{i=1}^{\infty} F_i \right\}$.

3 A priori bound for Leray equations

Recall that the profile $U(x)$ in (1.5) satisfies Leray equations (1.9) with zero boundary condition and $U(x) \rightarrow U_0(x)$ at spatial infinity. Decompose

$$U = U_0 + V.$$

The difference $V(x)$ satisfies

$$-\Delta V - V - x \cdot \nabla V + \nabla P = F_0 + F_1(V), \quad \operatorname{div} V = 0, \quad (3.1)$$

where

$$F_0(x) = \Delta U_0 + (U_0 + x \cdot \nabla U_0) - U_0 \cdot \nabla U_0, \quad (3.2)$$

$$F_1(V) = -(U_0 + V) \cdot \nabla V - V \cdot \nabla U_0, \quad (3.3)$$

and V vanishes at the boundary and the spatial infinity.

For a Sobolev function $f \in W^{1,2}(\Omega)$ put

$$\|f\|_{H^1(\Omega)} := \left(\int_{\Omega} |\nabla f|^2 + \frac{1}{2} |f|^2 \right)^{1/2}. \quad (3.4)$$

Denote by $H_0^1(\Omega)$ the closure of the set of all smooth functions having compact supports in Ω with respect to the norm $\|\cdot\|_{H^1(\Omega)}$, and

$$H_{0,\sigma}^1(\Omega) = \{f \in H_0^1(\Omega) : \operatorname{div} f = 0\}.$$

Note that $H_0^1(\Omega) = \{f \in W^{1,2}(\Omega) : f|_{\partial\Omega} = 0, \|f\|_{H^1(\Omega)} < \infty\}$ for bounded Lipschitz domains.

We assume the following.

Assumption 3.1 (Boundary data at infinity). *Let Ω be a domain in \mathbb{R}^3 . The vector fields $U_0 : \Omega \rightarrow \mathbb{R}^3$ and $F_0(x) = \Delta U_0 + (U_0 + x \cdot \nabla U_0) - U_0 \cdot \nabla U_0$ satisfy*

$$\|U_0\|_{L^6(\Omega)} < \infty, \quad \|\nabla U_0\|_{L^2(\Omega)} < \infty, \quad \left| \int_{\Omega} F_0 \cdot \eta \right| \leq C, \quad (3.5)$$

for any $\eta \in H_{0,\sigma}^1(\Omega)$ with $\|\eta\|_{H_{0,\sigma}^1(\Omega)} \leq 1$.

Note that from Assumption 3.1 it follows, in particular, that

$$\left| \int_{\Omega} (\eta \cdot \nabla) U_0 \cdot \eta \right| \leq C \quad (3.6)$$

for any $\eta \in H_{0,\sigma}^1(\Omega)$ with $\|\eta\|_{H_{0,\sigma}^1(\Omega)} \leq 1$ (by virtue of the evident imbedding $H_{0,\sigma}^1(\Omega) \hookrightarrow L^p$ for all $p \in [2, 6]$).

If it is valid in Ω , it is also valid in any subdomain of Ω with the same constant C . We will show in §5 that for $\Omega = \mathbb{R}_+^3$ and a satisfying (5.1), $U_0 = e^{-\frac{1}{2}A}a$ satisfies (5.3) and hence Assumption 3.1. This is also true if $\Omega = \mathbb{R}^3$ and a is self-similar, divergence free, and locally Hölder continuous.

Theorem 3.2 (A priori estimate for bounded domain). *Let Ω be a bounded domain in \mathbb{R}^3 with connected Lipschitz boundary $\partial\Omega$, and assume Assumption 3.1 for U_0 . Then for any function $V \in H_0^1(\Omega)$ satisfying*

$$-\Delta V + \nabla P = \lambda(V + x \cdot \nabla V + F_0 + F_1(V)), \quad \operatorname{div} V = 0, \quad (3.7)$$

with some $\lambda \in [0, 1]$, we have the a priori bound

$$\|V\|_{H^1(\Omega)}^2 = \int_{\Omega} \left(|\nabla V|^2 + \frac{1}{2}|V|^2 \right) \leq C(U_0, \Omega).$$

Remark. Note that $C(U_0, \Omega)$ is independent of $\lambda \in [0, 1]$.

Proof. Let the assumptions of the Theorem be fulfilled. Suppose that its assertion is not true. Then there exists a sequence of numbers $\lambda_k \in [0, 1]$ and functions $V_k \in H_0^1(\Omega)$ such that

$$-\Delta V_k - \lambda_k V_k - \lambda_k x \cdot \nabla V_k + \nabla P_k = \lambda_k (F_0 + F_1(V_k)), \quad \operatorname{div} V_k = 0, \quad (3.8)$$

moreover,

$$J_k^2 := \int_{\Omega} |\nabla V_k|^2 \rightarrow \infty. \quad (3.9)$$

Multiplying the equation (3.8) by V_k and integrating by parts in Ω , we obtain the identity

$$J_k^2 + \frac{\lambda_k}{2} \int_{\Omega} |V_k|^2 = \lambda_k \int_{\Omega} (F_0 - V_k \cdot \nabla U_0) V_k. \quad (3.10)$$

Consider the normalized sequence of functions

$$\widehat{V}_k = \frac{1}{J_k} V_k, \quad \widehat{P}_k = \frac{1}{\lambda_k J_k^2} P_k \quad (3.11)$$

Since

$$\int_{\Omega} |\nabla \widehat{V}_k|^2 \equiv 1,$$

we could extract a subsequence still denoted by \widehat{V}_k , which converges weakly in $W^{1,2}(\Omega)$ to some function $V \in H_0^1(\Omega)$, and strongly in $L^3(\Omega)$. Also we could assume without loss of generality that $\lambda_k \rightarrow \lambda_0 \in [0, 1]$.

Multiplying the identity (3.10) by $\frac{1}{J_k^2}$ and taking a limit as $k \rightarrow \infty$, we have

$$1 + \frac{\lambda_0}{2} \int_{\Omega} |V|^2 = -\lambda_0 \int_{\Omega} (V \cdot \nabla U_0) V = \lambda_0 \int_{\Omega} (V \cdot \nabla V) U_0. \quad (3.12)$$

In particular, λ_k is separated from zero for large k .

Multiplying the equation (3.8) by $\frac{1}{\lambda_k J_k^2}$, we see that the pairs $(\widehat{V}_k, \widehat{P}_k)$ satisfy the equation

$$\widehat{V}_k \cdot \nabla \widehat{V}_k + \nabla \widehat{P}_k = \frac{1}{J_k} \left(\frac{1}{\lambda_k} \Delta \widehat{V}_k + \widehat{V}_k + x \cdot \nabla \widehat{V}_k + F_0 - U_0 \cdot \nabla \widehat{V}_k - \widehat{V}_k \cdot \nabla U_0 \right). \quad (3.13)$$

Take arbitrary function $\eta \in C_{c,\sigma}^\infty(\Omega)$. Multiplying (3.13) by η , integrating by parts and taking a limit, we obtain finally

$$\int_{\Omega} (V \cdot \nabla V) \cdot \eta = 0. \quad (3.14)$$

Since η was arbitrary function from $C_{c,\sigma}^\infty(\Omega)$, we see that V is a weak solution to the Euler equation

$$\begin{cases} (V \cdot \nabla) V + \nabla P = 0 & \text{in } \Omega, \\ \operatorname{div} V = 0 & \text{in } \Omega, \\ V = 0 & \text{on } \partial\Omega, \end{cases} \quad (3.15)$$

with some $P \in D^{1,3/2}(\Omega) \cap L^3(\Omega)$. By Theorem 2.1, there exists a constant $\widehat{p}_0 \in \mathbb{R}$ such that $P(x) \equiv \widehat{p}_0$ on $\partial\Omega$. Of course, we can assume without loss of generality that $\widehat{p}_0 = 0$, i.e., $P(x) \equiv 0$ on $\partial\Omega$. Then by (3.12) and (3.15₁) we get

$$1 + \frac{\lambda_0}{2} \int_{\Omega} |V|^2 = -\lambda_0 \int_{\Omega} U_0 \cdot \nabla P = -\lambda_0 \int_{\Omega} \operatorname{div}(P \cdot U_0) = 0.$$

The obtained contradiction finishes the proof of the Theorem. \square

Theorem 3.3 (A priori bound for invading method). *Let $\Omega = \mathbb{R}_+^3$, and assume Assumption 3.1 for U_0 . Take a sequence of balls $B_k = B(0, R_k) \subset \mathbb{R}^3$ with $R_k \rightarrow \infty$, and consider half-balls $\Omega_k = \Omega \cap B_k$. Then for functions $V_k \in H_0^1(\Omega_k)$ satisfying*

$$-\Delta V_k - V_k - x \cdot \nabla V_k + \nabla P_k = F_0 + F_1(V_k), \quad \operatorname{div} V_k = 0, \quad (3.16)$$

we have the a priori bound

$$\int_{\Omega_k} \left(|\nabla V_k|^2 + \frac{1}{2} |V_k|^2 \right) \leq C(U_0)$$

where the constant $C(U_0)$ is independent of k .

Proof. Let the assumptions of the Theorem be fulfilled. Suppose that its assertion is not true. Then there exists a sequence of domains Ω_k and a sequence of solutions $V_k \in H_0^1(\Omega_k)$ of (3.16) such that

$$J_k^2 := \|V_k\|_{H^1(\Omega_k)}^2 = \int_{\Omega_k} \left(|\nabla V_k|^2 + \frac{1}{2} |V_k|^2 \right) \rightarrow \infty. \quad (3.17)$$

Multiplying the equation (3.16) by V_k and integrating by parts in Ω_k , we obtain the identity

$$J_k^2 = \int_{\Omega_k} (F_0 - V_k \cdot \nabla U_0) V_k. \quad (3.18)$$

Consider the normalized sequence of functions

$$\widehat{V}_k = \frac{1}{J_k} V_k, \quad \widehat{P}_k = \frac{1}{J_k^2} P_k \quad (3.19)$$

Multiplying the equation (3.16) by $\frac{1}{J_k^2}$, we see that the pairs $(\widehat{V}_k, \widehat{P}_k)$ satisfy the equation

$$\widehat{V}_k \cdot \nabla \widehat{V}_k + \nabla \widehat{P}_k = \frac{1}{J_k} (\Delta \widehat{V}_k + \widehat{V}_k + x \cdot \nabla \widehat{V}_k + F_0 - U_0 \cdot \nabla \widehat{V}_k - \widehat{V}_k \cdot \nabla U_0). \quad (3.20)$$

Since

$$\int_{\Omega_k} \left(|\nabla \widehat{V}_k|^2 + \frac{1}{2} |\widehat{V}_k|^2 \right) \equiv 1,$$

we could extract a subsequence still denoted by \widehat{V}_k , which converges weakly in $W^{1,2}(\Omega)$ to some function $V \in H_0^1(\Omega)$, and strongly in $L^2(E)$ for any $E \Subset \overline{\Omega}$.

Multiplying the identity (3.18) by $\frac{1}{J_k^2}$ and taking a limit as $k \rightarrow \infty$, we have

$$1 = \int_{\Omega} (-V \cdot \nabla U_0) V. \quad (3.21)$$

Take arbitrary function $\eta \in C_{c,\sigma}^\infty(\Omega)$. Multiplying (3.20) by η , integrating by parts and taking a limit, we obtain finally

$$\int_{\Omega} (V \cdot \nabla V) \cdot \eta = 0. \quad (3.22)$$

Since η was arbitrary function from $C_{c,\sigma}^\infty(\Omega)$, we see that V is a weak solution to the Euler equation

$$\begin{cases} (V \cdot \nabla) V + \nabla P &= 0 & \text{in } \Omega, \\ \operatorname{div} V &= 0 & \text{in } \Omega, \\ V &= 0 & \text{on } \partial\Omega, \end{cases} \quad (3.23)$$

with some $P \in D^{1,3/2}(\Omega) \cap L^3(\Omega)$. More precisely, since $V, \nabla V \in L^2(\Omega)$, we have $P \in D^{1,q}(\Omega)$ for every $q \in [1, 3/2]$, consequently, $P \in L^s(\Omega)$ for each $s \in [3/2, 3]$. From these inclusions, and using

$$\int_{S_R^+} |P|^q = -R^2 \int_R^\infty \int_{S_1^+} \frac{d}{dr} (|P(r\omega)|^q) d\omega dr \lesssim \int_{|x|>R} |P|^{q-1} |\nabla P|$$

where $S_R^+ = \{x \in \Omega : |x| = R\}$ is the corresponding half-sphere, it is easy to deduce that

$$\int_{S_R^+} |P|^q \rightarrow 0 \quad \text{as } R \rightarrow \infty$$

for every $q \in [1, 2]$. In particular,

$$\int_{S_R^+} |P|^{4/3} \rightarrow 0 \quad \text{as } R \rightarrow \infty. \quad (3.24)$$

Analogously, from the assumption $U_0 \in L^6(\Omega)$, $\nabla U \in L^2(\Omega)$ it is very easy to deduce that

$$\int_{S_R^+} |U_0|^4 \rightarrow 0 \quad \text{as } R \rightarrow \infty. \quad (3.25)$$

From the other hand, by (3.21) and (3.23₁) we obtain

$$1 = \int_{\Omega} (V \cdot \nabla) V \cdot U_0 = - \int_{\Omega} \nabla P \cdot U_0 = - \lim_{R \rightarrow \infty} \int_{\Omega_R} \operatorname{div}(P \cdot U_0) = - \lim_{R \rightarrow \infty} \int_{S_R^+} P(U_0 \cdot \mathbf{n}) = 0 \quad (3.26)$$

where $\Omega_R = \Omega \cap B(0, R)$ and the last equality follows from (3.24)–(3.25). The obtained contradiction finishes the proof of the Theorem. \square

4 Existence for Leray equations

The proof of existence theorem for the system of equations (3.1)–(3.3) in bounded domains Ω is based on the following fundamental fact.

Theorem 4.1 (Leray–Schauder Theorem). *Let $S : X \rightarrow X$ be a continuous and compact mapping of a Banach space X into itself, such that the set*

$$\{x \in X : x = \lambda Sx \quad \text{for some } \lambda \in [0, 1]\}$$

is bounded. Then S has a fixed point $x_ = Sx_*$.*

Let Ω be a domain in \mathbb{R}^3 with connected Lipschitz boundary $\Gamma = \partial\Omega$, and put $X = H_{0,\sigma}^1(\Omega)$.

For functions $V_1, V_2 \in H_{0,\sigma}^1(\Omega)$ denote $\langle V_1, V_2 \rangle_H = \int_{\Omega} \nabla V_1 \cdot \nabla V_2$. Then the system (3.1)–(3.3) is equivalent to the following identities:

$$\langle V, \zeta \rangle_H = \int_{\Omega} G(V) \cdot \zeta, \quad \forall \zeta \in C_{c,\sigma}^\infty(\Omega), \quad (4.1)$$

where $G(V) = V + x \cdot \nabla V + F(V)$, $F(V) = F_0 + F_1(V)$,

$$F_0(x) = \Delta U_0 + U_0 + x \cdot \nabla U_0 - U_0 \cdot \nabla U_0, \quad (4.2)$$

$$F_1(V) = -(U_0 + V) \cdot \nabla V - V \cdot \nabla U_0. \quad (4.3)$$

Since $H_{0,\sigma}^1(\Omega) \hookrightarrow L^6(\Omega)$, by Riesz representation theorem, for any $f \in L^{6/5}(\Omega)$ there exists a unique mapping $T(f) \in H_{0,\sigma}^1(\Omega)$ such that

$$\langle T(f), \zeta \rangle_H = \int_{\Omega} f \cdot \zeta, \quad \forall \zeta \in C_{c,\sigma}^\infty(\Omega), \quad (4.4)$$

moreover,

$$\|T(f)\|_H \leq \|f\|_{X'},$$

where

$$\|f\|_{X'} = \sup_{\zeta \in C_{c,\sigma}^\infty(\Omega), \|\zeta\|_H \leq 1} \int_{\Omega} f \cdot \zeta.$$

Then the system (3.1)–(3.3)~(4.1) is equivalent to the equality

$$V = T(G(V)). \quad (4.5)$$

Theorem 4.2 (Compactness). *If Ω is a bounded domain in \mathbb{R}^3 with connected Lipschitz boundary $\Gamma = \partial\Omega$, and Assumption 3.1 holds for U_0 , then for $X = H_{0,\sigma}^1(\Omega)$ the operator $S : X \ni V \mapsto T(G(V)) \in X$ is continuous and compact.*

Proof. (i) For $V, \tilde{V} \in X$, denoting $v = \tilde{V} - V$,

$$F(\tilde{V}) - F(V) = -(U_0 + V + v) \cdot \nabla v - v \cdot \nabla(U_0 + V).$$

Thus we have

$$\begin{aligned} \|S(\tilde{V}) - S(V)\|_X &\lesssim \|v\|_{L^2} + \|\nabla v\|_{L^2} + \|F(\tilde{V}) - F(V)\|_{L^2 + L^{6/5}} \\ &\lesssim \|v\|_{L^2} + \|\nabla v\|_{L^2} + \|U_0\|_{L^2} \|\nabla v\|_{L^2} + \|V + v\|_{L^2} \|\nabla v\|_{L^2} + \|\nabla U_0\|_{L^2} \|v\|_{L^2} + \|v\|_{L^2} \|\nabla V\|_{L^2} \\ &\lesssim (1 + \|V\|_X + \|v\|_X) \|v\|_X. \end{aligned} \quad (4.6)$$

(ii) By Sobolev Theorems, we have the compact embedding: $X \hookrightarrow L^r(\Omega) \quad \forall r \in [1, 6)$. Thus if a sequence $V_k \in X$ is bounded in X , i.e., $\|V_k\|_{L^2(\Omega)} + \|\nabla V_k\|_{L^2(\Omega)} \leq C$, then we can extract a subsequence V_{k_l} which converges to some $V \in X$ in $L^3(\Omega)$ norm: $\|V_{k_l} - V\|_{L^3(\Omega)} \rightarrow 0$ as $l \rightarrow \infty$. Then using the condition $V_{k_l} \equiv V \equiv 0$ on $\partial\Omega$ and integration by parts, it is easy to see that $\|F(V_{k_l}) - F(V)\|_{X'} \rightarrow 0$ and, consequently, $\|G(V_{k_l}) - G(V)\|_{X'} \rightarrow 0$ as $l \rightarrow \infty$. \square

Corollary 4.3 (Existence in bounded domains). *Let Ω be a bounded domain in \mathbb{R}^3 with connected Lipschitz boundary $\partial\Omega$, and assume Assumption 3.1 for U_0 . Then the system (3.1)–(3.3) has a solution $V \in H_{0,\sigma}^1(\Omega)$.*

Proof. This is a direct consequence of Theorems 4.1–4.2 and 3.2. \square

Theorem 4.4 (Existence in unbounded domains). *Let $\Omega = \mathbb{R}_+^3$, and assume Assumption 3.1 for U_0 . Then the system (3.1)–(3.3) has a solution $V \in H_{0,\sigma}^1(\Omega)$.*

Proof. Take balls $B_k = B(0, k)$ and consider the increasing sequence of domains $\Omega_k = \Omega \cap B_k$ from Theorem 3.3. By Corollary 4.3 there exists a sequence of solutions $V_k \in H_{0,\sigma}^1(\Omega_k)$ of the system (3.1)–(3.3) in Ω_k . By Theorem 3.3, the norms $\|V_k\|_{H_{0,\sigma}^1(\Omega)}$ are uniformly bounded, thus we can extract a subsequence V_{k_l} such that the weak convergence $V_{k_l} \rightharpoonup V$ in $W^{1,2}(\Omega')$ holds for any bounded subdomain $\Omega' \subset \Omega$. It is easy to check that the limit function V is a solution of the system (3.1)–(3.3) in Ω . \square

5 Boundary data at infinity in the half space

In this section we restrict ourselves to the half space $\Omega = \mathbb{R}_+^3$ with boundary $\Sigma = \partial\mathbb{R}_+^3$ and study the decay property of $U_0 = e^{-\frac{1}{2}A}a$. Our goal is to prove the following lemma, which ensures Assumption 3.1 under the conditions of Theorem 1.1.

Denote $x^* = (x', -x_3)$ for $x = (x', x_3) \in \mathbb{R}^3$, and $\langle z \rangle = (1 + |z|^2)^{1/2}$ for $z \in \mathbb{R}^m$.

Lemma 5.1. *Suppose a is a vector field in $\Omega = \mathbb{R}_+^3$ satisfying*

$$\begin{aligned} a &\in C_{loc}^1(\bar{\Omega} \setminus \{0\}; \mathbb{R}^3), \quad \operatorname{div} a = 0, \quad a|_{\partial\Omega} = 0, \\ a(x) &= \lambda a(\lambda x) \quad \forall x \in \Omega, \quad \forall \lambda > 0. \end{aligned} \quad (5.1)$$

Let $U_0 = e^{-\frac{1}{2}A}a$, where A is the Stokes operator in Ω . Then

$$|\nabla^k U_0(x)| \leq c_k [a]_1 (1 + x_3)^{-\min(1,k)} (1 + |x|)^{-1}, \quad \forall k \in \mathbb{Z}_+ = \{0, 1, 2, \dots\}, \quad (5.2)$$

and, for any $0 < \delta \ll 1$,

$$|\nabla U_0(x)| + |U_0(x) + x \cdot \nabla U_0(x)| \leq c_\delta [a]_1 x_3^{-\delta} \langle x \rangle^{2\delta-2}, \quad (5.3)$$

where $[a]_m = \sup_{k \leq m, |x|=1} |\nabla^k a(x)|$.

If we further assume $a \in C_{loc}^m$, $m \geq 2$, and $\partial_3^k a|_\Sigma = 0$ for $k < m$, then $|\nabla^k U_0(x)| \leq c_k [a]_m \langle x_3 \rangle^{-k} \langle x \rangle^{-1}$ for $k \leq m$.

Estimates (5.2) and (5.3) imply, in particular,

$$U_0 \in L^4(\Omega) \cap L^\infty(\Omega), \quad \nabla U_0 \in L^2(\Omega), \quad (U_0 + x \cdot \nabla U_0) \in L^2(\Omega), \quad (5.4)$$

and hence Assumption 3.1 for U_0 is satisfied.

5.1 Green tensor for nonstationary Stokes system in half space

Consider the nonstationary Stokes system in half space \mathbb{R}_+^3 ,

$$\partial_t v - \Delta v + \nabla p = 0, \quad \operatorname{div} v = 0, \quad (x \in \mathbb{R}_+^3, \quad t > 0), \quad (5.5)$$

$$v|_{x_3=0} = 0, \quad v|_{t=0} = a. \quad (5.6)$$

It is shown by Solonnikov [22, §2] that, if $a = \check{a}$ satisfies

$$\operatorname{div} \check{a} = 0, \quad \check{a}_3|_{x_3=0} = 0, \quad (5.7)$$

then

$$v_i(x, t) = \int_{\mathbb{R}_+^3} \check{G}_{ij}(x, y, t) \check{a}_j(y) dy \quad (5.8)$$

with

$$\begin{aligned} \check{G}_{ij}(x, y, t) &= \delta_{ij} \Gamma(x - y, t) + G_{ij}^*(x, y, t) \\ G_{ij}^*(x, y, t) &= -\delta_{ij} \Gamma(x - y^*, t) \\ &\quad - 4(1 - \delta_{jn}) \frac{\partial}{\partial x_j} \int_{\mathbb{R}^2 \times [0, x_3]} \frac{\partial}{\partial x_i} E(x - z) \Gamma(z - y^*, t) dz, \end{aligned} \quad (5.9)$$

where $E(x) = \frac{1}{4\pi|x|}$ and $\Gamma(x, t) = (4\pi t)^{-3/2} e^{-\frac{|x|^2}{4t}}$ are the fundamental solutions of the Laplace and heat equations in \mathbb{R}^3 . (A sign difference occurs since $E(x) = \frac{-1}{4\pi|x|}$ in [22].) Moreover, G_{ij}^* satisfies the pointwise bound ([22, (2.38)])

$$|\partial_t^s D_x^k D_y^\ell G_{ij}^*(x, y, t)| \lesssim t^{-s-\ell_3/2} (\sqrt{t} + x_3)^{-k_3} (\sqrt{t} + |x - y^*|)^{-3-|k'| - |\ell'|} e^{-\frac{cy_3^2}{t}} \quad (5.10)$$

for all $s \in \mathbb{N} = \{0, 1, 2, \dots\}$ and $k, \ell \in \mathbb{N}^3$.

Note that \check{G}_{ij} is not the Green tensor in the strict sense since it requires (5.7). There is no known pointwise estimate for the Green tensor, cf. Solonnikov [21] and Kang [8].

We now estimate $U_0 = e^{-\frac{1}{2}A}a$ for a satisfying (5.1). By (5.8) and (5.9),

$$U_{0,i}(x) = \int_{\mathbb{R}_+^3} \Gamma(x - y, \tfrac{1}{2}) a_i(y) dy + \int_{\mathbb{R}_+^3} G_{ij}^*(x, y, \tfrac{1}{2}) a_j(y) dy =: U_{1,i}(x) + U_{2,i}(x). \quad (5.11)$$

By (5.10), for $k \in \mathbb{Z}_+$ and using only $|a(y)| \lesssim \frac{1}{|y'|}$,

$$\begin{aligned} |\nabla^k U_2(x)| &\lesssim \int_{\mathbb{R}_+^3} (1 + x_3)^{-k} (1 + x_3 + |x' - y'|)^{-3} e^{-cy_3^2} \frac{1}{|y'|} dy \\ &\lesssim (1 + x_3)^{-k} \int_{\mathbb{R}^2} (1 + x_3 + |x' - y'|)^{-3} \frac{1}{|y'|} dy' \\ &= (1 + x_3)^{-k-2} \int_{\mathbb{R}^2} (1 + |\bar{x} - z'|)^{-3} \frac{1}{|z'|} dz', \quad (\bar{x} = \frac{x'}{1 + x_3}) \\ &\lesssim (1 + x_3)^{-k-2} (1 + |\bar{x}|)^{-1} \\ &= (1 + x_3)^{-k-1} (1 + x_3 + |x'|)^{-1}. \end{aligned} \quad (5.12)$$

To estimate U_1 , fix a cut-off function $\zeta(x) \in C_c^\infty(\mathbb{R}^3)$ with $\zeta(x) = 1$ for $|x| < 1$. We have

$$\nabla^k U_{1,i}(x) = \int_{\mathbb{R}_+^3} \Gamma(x - y, \tfrac{1}{2}) \nabla_y^k ((1 - \zeta(y)) a_i(y)) dy + \int_{\mathbb{R}_+^3} \nabla_x^k \Gamma(x - y, \tfrac{1}{2}) (\zeta(y) a_i(y)) dy \quad (5.13)$$

where we used $a|_\Sigma = 0$, and hence, for $k \leq 1$,

$$|\nabla^k U_1(x)| \lesssim \int_{\mathbb{R}^3} e^{-|x-y|^2/2} \langle y \rangle^{-1-k} dy + e^{-x^2/4} \lesssim \langle x \rangle^{-1-k}. \quad (5.14)$$

We can get the same estimate for $k \geq 2$ if we assume $\nabla^k a$ is defined and has the same decay. On the other hand, we can show $|\nabla_x^k U_1(x)| \lesssim \langle x \rangle^{-2}$ for $k \geq 2$ if we place the extra derivatives on Γ in the first integral of (5.13).

Combining (5.12) and (5.14), we get (5.2) and the last statement of Lemma 5.1.

Also,

$$\begin{aligned} U_1(x) + x \cdot \nabla U_1(x) &= \int_{\mathbb{R}_+^3} \Gamma(y, \tfrac{1}{2}) (1 + (x - y) \cdot \nabla_x) a_i(x - y) dy \\ &\quad + \int_{\mathbb{R}_+^3} \Gamma(y, \tfrac{1}{2}) y \cdot \nabla_x a_i(x - y) dy. \end{aligned}$$

Since a is minus one homogeneous, the first integral is zero. Using $se^{-s^2} \leq C_k(1 + s^2)^{-k}$ for any $k \geq 0$ and $s \geq 0$, and choosing $k = 2$ and $s = |y|/\sqrt{4t}$,

$$|U_1(x) + x \cdot \nabla U_1(x)| \lesssim \int_{\mathbb{R}^3} (1 + y^2)^{-2} |x - y|^{-2} dy \lesssim (x^2 + 1)^{-1}. \quad (5.15)$$

Denote

$$\Omega_- = \{x \in \Omega : 1 + x_3 > |x'|\}, \quad \Omega_+ = \{x \in \Omega : 1 + x_3 \leq |x'|\}. \quad (5.16)$$

By (5.12), (5.14), and (5.15), we have shown (5.3) in Ω_- (with $\delta = 0$).

It remains to show (5.3) in Ω_+ .

5.2 Estimates using boundary layer integral

Denote $\varepsilon_j = 1$ for $j < 3$ and $\varepsilon_3 = -1$. Thus $x_j^* = \varepsilon_j x_j$. Let $\bar{a}(x)$ be an extension of $a(x)$ to $x \in \mathbb{R}^3$ with

$$\bar{a}_j(x) = \varepsilon_j a_j(x^*), \quad \text{if } x_3 < 0.$$

Since $\operatorname{div} a = 0$ in \mathbb{R}_+^3 and $a|_\Sigma = 0$, it follows that $\operatorname{div} \bar{a} = 0$ in \mathbb{R}^3 . Let $u(x, t)$ be the solution of the nonstationary Stokes system in \mathbb{R}^3 with initial data \bar{a} , given simply by

$$u_i(x, t) = \int_{\mathbb{R}^3} \Gamma(y, t) \bar{a}_i(x - y) dy.$$

It follows that $u_i(x, t) = \varepsilon_i u_i(x^*, t)$. Thus

$$\partial_3 u_i(x, t)|_\Sigma = 0, \quad (i < 3); \quad u_3(x, t)|_\Sigma = 0. \quad (5.17)$$

We have $|\nabla^k a(y)| \lesssim |y|^{-1-k}$ for $k \leq 1$. By the same estimates leading to (5.14) and (5.15) for U_1 , we have

$$|\nabla_x^k u_i(x, \frac{1}{2})| \lesssim \langle x \rangle^{-1-\min(1,k)}, \quad (k \leq 2), \quad (5.18)$$

and

$$|(u + x \cdot \nabla u)(x, \frac{1}{2})| \lesssim (|x|^2 + 1)^{-1}. \quad (5.19)$$

Thus $u(x, \frac{1}{2})$ satisfies (5.3).

Using self-similarity condition

$$u(x, t) = \lambda u(\lambda x, \lambda^2 t) \quad \forall \lambda > 0, \quad (5.20)$$

from (5.18) and (5.19) we get

$$|\nabla_x^m u_i(x, t)| \lesssim \begin{cases} (|x| + \sqrt{t})^{-1-m}, & (m \leq 1), \\ t^{-1/2} (|x| + \sqrt{t})^{-2}, & (m = 2). \end{cases} \quad (5.21)$$

$$|(u + x \cdot \nabla u)(x, t)| \lesssim \sqrt{t} (|x|^2 + t)^{-1}. \quad (5.22)$$

Decompose now

$$v = u - w.$$

Then w satisfies the nonstationary Stokes system in \mathbb{R}_+^3 with zero force, zero initial data, and has boundary value

$$w_j(x, t)|_{x_3=0} = u_j(x', 0, t), \quad \text{if } j < 3; \quad w_3(x, t)|_{x_3=0} = 0. \quad (5.23)$$

It is given by the boundary layer integral (using (5.23)),

$$w_i(x, t) = \sum_{j=1,2} \int_0^t \int_\Sigma K_{ij}(x - z', s) u_j(z', 0, t - s) dz' ds, \quad (5.24)$$

where, for $j < 3$, ([21, pp. 40, 48])

$$K_{ij}(x, t) = -2\delta_{ij}\partial_3\Gamma - \frac{1}{\pi}\partial_j\mathcal{C}_i, \quad (5.25)$$

$$\mathcal{C}_i(x, t) = \int_{\Sigma \times [0, x_3]} \partial_3\Gamma(y, t) \frac{y_i - x_i}{|y - x|^3} dy. \quad (5.26)$$

(Note that K_{i3} ($j = 3$) have extra terms.) They satisfy for $j < 3$ ([21, pp. 41, 48])

$$|\partial_t^m D_{x'}^\ell \partial_{x_3}^k \mathcal{C}_i(x, t)| \leq ct^{-m-\frac{1}{2}}(x_3 + \sqrt{t})^{-k}(|x| + \sqrt{t})^{-2-\ell}. \quad (5.27)$$

We now show (5.3) for $w(x, 1/2)$ in the region $\Omega_+ : 1 + x_3 \leq |x'|$.

For $t = 1/2$ and $i, k \in \{1, 2, 3\}$,

$$\begin{aligned} \partial_{x_k} w_i(x, \tfrac{1}{2}) &= - \sum_{j=1,2} \int_0^{\frac{1}{2}} \int_{\Sigma} \frac{1}{\pi} \partial_k \mathcal{C}_i(x - z', s) \partial_{z_j} u_j(z', 0, \tfrac{1}{2} - s) dz' ds \\ &\quad - \mathbf{1}_{i < 3} \int_0^{\frac{1}{2}} \int_{\Sigma} 2\partial_k \partial_3 \Gamma(x - z', s) u_i(z', 0, \tfrac{1}{2} - s) dz' ds \\ &= I_1 + I_2. \end{aligned} \quad (5.28)$$

Above, we have integrated by parts in tangential directions x_j in I_1 .

By (5.21) and (5.27),

$$|I_1| \lesssim \int_0^{\frac{1}{2}} \int_{\Sigma} s^{-\frac{1}{2}} (x_3 + \sqrt{s})^{-1} (|x - z'| + \sqrt{s})^{-2} (|z'| + \sqrt{\tfrac{1}{2} - s})^{-2} dz' ds.$$

Fix $0 < \varepsilon \leq 1/2$. Splitting $(0, 1/2) = (0, 1/4] \cup (1/4, 1/2)$, and changing variable $s \rightarrow 1/2 - s$ in $(1/4, 1/2)$, we get

$$\begin{aligned} |I_1| &\lesssim \int_0^{\frac{1}{4}} \int_{\Sigma} x_3^{-2\varepsilon} s^{-1+\varepsilon} (|x' - z'| + x_3 + \sqrt{s})^{-2} (|z'| + 1)^{-2} dz' ds \\ &\quad + \int_0^{\frac{1}{4}} \int_{\Sigma} (x_3 + 1)^{-1} (|x' - z'| + x_3 + 1)^{-2} (|z'| + \sqrt{s})^{-2} dz' ds. \end{aligned}$$

Integrating first in time and using that, for $0 < b < \infty$, $0 \leq a < 1 < a + b$, and $0 < N < \infty$,

$$\int_0^1 \frac{ds}{s^a (N + s)^b} \leq \frac{C}{N^{a+b-1} (N + 1)^{1-a}}, \quad (5.29)$$

$$\int_0^1 \frac{ds}{s^a (N + s)^{1-a}} \leq C \min \left(\frac{1}{N^{1-a}}, \log \frac{2N + 2}{N} \right), \quad (5.30)$$

where the constant C is independent of N , we get

$$\begin{aligned} |I_1| &\lesssim \int_{\Sigma} x_3^{-2\varepsilon} (|x' - z'| + x_3)^{-2+2\varepsilon} (|x' - z'| + x_3 + 1)^{-2\varepsilon} (|z'| + 1)^{-2} dz' \\ &\quad + \int_{\Sigma} (x_3 + 1)^{-1} (|x' - z'| + x_3 + 1)^{-2} \min \left(\frac{1}{|z'|^2}, \log \frac{2|z'|^2 + 2}{|z'|^2} \right) dz'. \end{aligned}$$

Dividing the integration domain to $|z'| < |x'|/2$, $|x'|/2 < |z'| < 2|x'|$ and $|z'| > 2|x'|$, we get

$$|I_1| \lesssim x_3^{-2\varepsilon} \langle x \rangle^{-2+\delta}, \quad (x \in \Omega_+) \quad (5.31)$$

for any $0 < \delta \ll 1$. Taking $\varepsilon = \delta/2$ and $\varepsilon = 1/2$, we get

$$(1 + x_3)|I_1| \lesssim x_3^{-\delta} \langle x \rangle^{-2+2\delta}, \quad (x \in \Omega_+). \quad (5.32)$$

To estimate I_2 for $i < 3$ (note $I_2 = 0$ if $i = 3$), we separate two cases. If $k < 3$, integration by parts gives

$$I_2 = - \int_0^{\frac{1}{2}} \int_{\Sigma} 2\partial_3 \Gamma(x - z', s) \partial_{z_k} u_i(z', 0, \frac{1}{2} - s) dz' ds.$$

Using $ue^{-u^2} \leq C_\ell(1+u)^{-\ell}$ for $u > 0$ for any $\ell > 0$,

$$\partial_3 \Gamma(x, s) = cs^{-2} \frac{x_3}{\sqrt{s}} e^{-x^2/4s} \leq cs^{-2} (1 + \frac{|x|}{\sqrt{s}})^{-3} = cs^{-1/2} (|x| + \sqrt{s})^{-3}. \quad (5.33)$$

Hence I_2 can be estimated in the same way as I_1 , and (5.32) is valid if I_1 is replaced by I_2 and $k < 3$.

When $k = 3$, by $\partial_t \Gamma = \Delta \Gamma$ and integration by parts,

$$\begin{aligned} I_2 &= \int_0^{\frac{1}{2}} \int_{\Sigma} 2(\sum_{j<3} \partial_j^2 - \partial_t) \Gamma(x - z', s) u_i(z', 0, \frac{1}{2} - s) dz' ds \\ &= \sum_{j<3} \int_0^{\frac{1}{2}} \int_{\Sigma} 2\partial_j \Gamma(x - z', s) \partial_{z_j} u_i(z', 0, \frac{1}{2} - s) dz' ds \\ &\quad + \int_0^{\frac{1}{2}} \int_{\Sigma} 2\Gamma(x - z', s) \partial_t u_i(z', 0, \frac{1}{2} - s) dz' ds \\ &\quad - \lim_{\mu \rightarrow 0_+} \left(\int_{\Sigma} 2\Gamma(x - z', \frac{1}{2} - \mu) u_i(z', 0, \mu) dz - \int_{\Sigma} 2\Gamma(x - z', \mu) u_i(z', 0, \frac{1}{2} - \mu) dz \right) \\ &= I_3 + I_4 + \lim_{\mu \rightarrow 0_+} (I_{5,\mu} + I_{6,\mu}). \end{aligned}$$

Here I_3 can be estimated in the same way as I_1 , and (5.32) is valid if I_1 is replaced by I_3 . For I_4 , since $\partial_t u_i = \Delta u_i$, by estimate (5.21) for $\nabla^2 u$,

$$|I_4| \lesssim \int_0^{\frac{1}{2}} \int_{\Sigma} s^{-\frac{3}{2}} (1 + \frac{|x - z'|^2}{4s})^{-\frac{3}{2}} (\frac{1}{2} - s)^{-\frac{1}{2}} (|z'| + \sqrt{\frac{1}{2} - s})^{-2} dz' ds. \quad (5.34)$$

We have similar estimate as I_1 with the following difference: we have to use the estimate (5.29) during the integration over each subinterval $s \in [0, 1/4]$ and $s \in [1/4, 1/2]$, for the second subinterval we apply (5.29) with $a = \frac{1}{2}$, $b = 1$, $N = |z'|^2$.

For the boundary terms, the integrand of $I_{5,\mu}$ is bounded by $e^{-\frac{|x-z'|^2}{2}} |z'|^{-1}$ and converges to 0 as $\mu \rightarrow 0_+$ for each $z' \in \Sigma$. Thus $\lim I_{5,\mu} = 0$ by Lebesgue dominated convergence theorem. For $I_{6,\mu}$,

$$|I_{6,\mu}| \lesssim \mu^{-1/2} e^{-\frac{x_3^2}{4\mu}} \int_{\Sigma} \Gamma_{\mathbb{R}^2}(x' - z', \mu) \frac{1}{\langle z' \rangle} dz' \lesssim \mu^{-1/2} e^{-\frac{x_3^2}{4\mu}} \frac{1}{\langle x' \rangle}, \quad (5.35)$$

which converges to 0 as $\mu \rightarrow 0_+$ for any $x \in \Omega$.

We conclude that, for either $k < 3$ or $k = 3$, (5.32) is valid if I_1 is replaced by I_2 and hence, for any $0 < \delta \ll 1$,

$$(1 + x_3)|\partial_k w_i(x, \frac{1}{2})| \lesssim x_3^{-\delta} \langle x \rangle^{-2+2\delta}, \quad \forall x \in \Omega_+, \quad \forall i, k \leq 3. \quad (5.36)$$

It remains to estimate $(1 + \sum_{k < 3} x_k \partial_k)w$. For $k < 3$,

$$\begin{aligned} x_k \partial_3 \Gamma(x - z', s) &= z_k \partial_3 \Gamma(x - z', s) + 2s \partial_k \partial_3 \Gamma(x - z', s), \\ x_k \mathcal{C}_i(x - z', s) &= z_k \mathcal{C}_i(x - z', s) + 2s \partial_k \mathcal{C}_i(x - z', s), \end{aligned}$$

by the explicit formula (5.26) of \mathcal{C}_i . Thus, by (5.28) and integration by parts,

$$\begin{aligned} &(1 + \sum_{k < 3} x_k \partial_k)w_i(x, \frac{1}{2}) \\ &= - \sum_{j=1,2} \int_0^{\frac{1}{2}} \int_{\Sigma} \frac{1}{\pi} \mathcal{C}_i(x - z', s) (1 + \sum_{k < 3} z_k \partial_{z_k}) \partial_{z_j} u_j(z', 0, \frac{1}{2} - s) dz' ds \\ &\quad - \mathbf{1}_{i < 3} \int_0^{\frac{1}{2}} \int_{\Sigma} 2 \partial_3 \Gamma(x - z', s) (1 + \sum_{k < 3} z_k \partial_{z_k}) u_i(z', 0, \frac{1}{2} - s) dz' ds \\ &\quad - \sum_{j,k < 3} \int_0^{\frac{1}{2}} \int_{\Sigma} \frac{1}{\pi} 2s \partial_k \mathcal{C}_i(x - z', s) \partial_k \partial_{z_j} u_j(z', 0, \frac{1}{2} - s) dz' ds \\ &\quad - \mathbf{1}_{i < 3} \sum_{k < 3} \int_0^{\frac{1}{2}} \int_{\Sigma} 4s \partial_k \partial_3 \Gamma(x - z', s) \partial_k u_i(z', 0, \frac{1}{2} - s) dz' ds \\ &= I_7 + I_8 + I_9 + I_{10}. \end{aligned}$$

For I_8 , note $\sum_{k < 3} z_k \partial_{z_k} = r \partial_r$ on Σ and hence

$$|(1 + \sum_{k < 3} z_k \partial_{z_k}) u_i(z', 0, t - s)| \leq \sqrt{t - s} (|z'|^2 + t - s)^{-1} \quad (5.37)$$

by (5.22). In particular the second factor of the integrand has more decay than $|z'|^{-1}$. Thus I_8 can be estimated similarly as I_1 (see also (5.33)).

For I_7 , noting $(1 + \sum_{k < 3} z_k \partial_{z_k}) \partial_{z_j} u_j = \partial_{z_j} (1 + \sum_{k < 3} z_k \partial_{z_k}) u_j - \partial_j u_j$, we have

$$|I_7| \lesssim \sum_{j < 3} \int_0^{\frac{1}{2}} \int_{\Sigma} |\partial_j \mathcal{C}_i(x - z', s)| (5.37) + |\mathcal{C}_i(x - z', s)| (|z'|^2 + \frac{1}{2} - s)^{-1} dz' ds$$

and can be estimated similarly as I_1 . Finally, both I_9 and I_{10} can be estimated similarly as I_1 . Here we have used the estimates

$$s |\partial_x^\ell C(x - z', s)| \leq s^{1/2} (|x - z'| + \sqrt{s})^{-2-\ell},$$

see (5.27), and

$$|s \partial_k \partial_3 \Gamma(x, s)| = cs^{-5/2} |x_3 x_k| e^{-x^2/4s} \leq cs^{-3/2} \frac{x^2}{s} e^{-x^2/4s} \lesssim s^{-3/2} \left(\frac{|x|}{\sqrt{s}} + 1 \right)^{-3} = (|x| + \sqrt{s})^{-3}.$$

Summarizing, we have

$$|(1 + \sum_{k < 3} x_k \partial_k)w_i(x, \frac{1}{2})| \lesssim x_3^{-\delta} \langle x \rangle^{-2+2\delta}, \quad \forall x \in \Omega_+, \quad \forall i \leq 3. \quad (5.38)$$

Combining (5.18), (5.19), (5.36) and (5.38), we have shown (5.3) in Ω_+ . This concludes the proof of Lemma 5.1. \square

6 Self-similar solutions in the half space

In this section we first complete the proof of Theorem 1.1, and then give a few comments.

Proof of Theorem 1.1. By Lemma 5.1, for those a satisfying the assumptions of Theorem 1.1, $U_0 = e^{-\frac{1}{2}A}a$ satisfies (5.2) and (5.3), and hence Assumption 3.1 is satisfied. By Theorem 4.4, there is a solution $V \in H_{0,\sigma}^1(\mathbb{R}_+^3)$ of the system (3.1)–(3.3).

Noting $U_0 \in C^\infty(\mathbb{R}_+^3)$ by (5.2), the system (3.1)–(3.3) is a perturbation of the stationary Navier-Stokes system with smooth coefficients. The regularity theory for Navier-Stokes system implies that $V \in C_{loc}^\infty(\mathbb{R}_+^3)$. The vector field $U = U_0 + V$ is thus a smooth solution of the Leray equations (1.9) in \mathbb{R}_+^3 .

The vector field $u(x, t)$ defined by (1.5), $u(x, t) = \frac{1}{\sqrt{2t}} U\left(\frac{x}{\sqrt{2t}}\right)$, is thus smooth and self-similar. Moreover,

$$v(x, t) = u(x, t) - e^{-tA}a = \frac{1}{\sqrt{2t}} V\left(\frac{x}{\sqrt{2t}}\right)$$

satisfies $\|v(t)\|_{L^q(\mathbb{R}_+^3)} = \|V\|_{L^q(\mathbb{R}_+^3)}(2t)^{\frac{3}{2q}-\frac{1}{2}}$ and $\|\nabla v(t)\|_{L^2(\mathbb{R}_+^3)} = \|\nabla V\|_{L^2(\mathbb{R}_+^3)}(2t)^{-1/4}$. This finishes the proof of Theorem 1.1. \square

Remark. Let $u_0(x, t) = (e^{-tA}a)(x) = \frac{1}{\sqrt{2t}} U_0\left(\frac{x}{\sqrt{2t}}\right)$. We have $u_0(\cdot, t) \rightarrow a$ as $t \rightarrow 0_+$ in $L^{3,\infty}(\mathbb{R}_+^3)$. Indeed, by (5.2), $|U_0(x)| \lesssim \langle x \rangle^{-1} \in L^{3,\infty} \cap L^q$, $q > 3$. We have $\|u_0(t)\|_{L^q(\mathbb{R}_+^3)} = \|U_0\|_{L^q(\mathbb{R}_+^3)}(2t)^{\frac{3}{2q}-\frac{1}{2}}$, which remains finite as $t \rightarrow 0_+$ only if $q = (3, \infty)$, and

$$|u_0(x, t)| \lesssim \frac{1}{\sqrt{t}} \cdot \frac{1}{1 + \frac{|x|}{\sqrt{t}}} = \frac{1}{\sqrt{t} + |x|}. \quad (6.1)$$

This is consistent with the whole space case $\Omega = \mathbb{R}^3$.

For the difference $V(x)$, we only have its $L^q(\mathbb{R}_+^3)$ bounds, and not pointwise bounds as (1.11) in [7, 24].

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